MATH 4030 Differential Geometry
Tutorial 6, 18 October 2017

1. Find the mean curvature H and the Gauss curvature K of the following surfaces:
   (a) $S_1 = \{(R\cos\theta, R\sin\theta, z)|\ \theta \in [0, 2\pi], z \in \mathbb{R}\}$ (cylinder)
   (b) $S_2 = \{(x, y, x^2 - y^2)|\ x, y \in \mathbb{R}\}$ (hyperbolic paraboloid)

   where the normal is taken to be the outward normal for (a) and the upward normal for (b).

Solution.

(a) Let $X : I \times \mathbb{R} \to S_1 : (\theta, z) \mapsto (R\cos\theta, R\sin\theta, z)$ where I is $(0, 2\pi)$ or $(-\pi, \pi)$. Then X is a parametrization of S_1 and

$$\begin{aligned} X_\theta &= (-R\sin\theta, R\cos\theta, 0) \\ X_z &= (0, 0, 1) \\ X_\theta \times X_z &= (R\cos\theta, R\sin\theta, 0) \\ N = \frac{X_\theta \times X_z}{|X_\theta \times X_z|} &= (\cos\theta, \sin\theta, 0) \end{aligned}$$

Note that N agrees with the given normal vector. Let $\mathcal{S}$ be the shape operator. Then we have

$$\begin{aligned} S(X_\theta) &:= -\frac{\partial N}{\partial\theta} \\ &= (\sin\theta, -\cos\theta, 0) \\ &= -\frac{1}{R} \cdot X_\theta + 0 \cdot X_z; \end{aligned}$$

and

$$\begin{aligned} S(X_z) &:= -\frac{\partial N}{\partial z} \\ &= (0, 0, 0) \\ &= 0 \cdot X_\theta + 0 \cdot X_z. \end{aligned}$$

It follows that

$$[\mathcal{S}]_{\{X_\theta, X_z\}} = \begin{pmatrix} -\frac{1}{R} & 0 \\ 0 & 0 \end{pmatrix}.$$

What we want to find are

$$H = \mathrm{tr}(\mathcal{S}) = -\frac{1}{R} \text{ and } K = \det(\mathcal{S}) = 0.$$

(b) See Solutions to HW3 Q7.

2. Prove the spectral theorem for 3×3 symmetric real matrices: let $A \in \mathcal{M}_{3\times3}(\mathbb{R})$ with $A^T = A$. Then $\mathbb{R}^3$ has an orthonormal basis consisting of eigenvectors of A.

Solution. See p.18-19 of Lecture notes (part 3).